GOVERNMENT OF ANDHRA PRADESH

COMMISSIONERATE OF COLLEGIATE EDUCATION

INNER PRODUCT SPACES

ORTHONORMAL SET

MATHEMATICS

K.SAMRAJYAM
M.Sc, B.Ed, M.Phil

P.R.GOVT.COLLEGE(A):KAKINADA

Email ID :tptswamy82@gmail.com

LEARNING OBJECTIVES

The learning objectives are

- ❖ Gets the knowledge on the inner product spaces.
- ❖ Understands the concept of orthonormal set in an inner product spaces.
- ❖ Analysis the concept of inner product spaces.
- ❖ Finds the orthonormal set in an inner product spaces.
- ❖ Verifies the orthonormality to the given set

INNER PRODUCT SPACES

ORTHONORMAL SET

DEFINITION: Let S be a non-empty subset of an inner product space V(F). The set S={$\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ }is said to be an ortho normal set if

(1) $\langle \alpha_{i,} \alpha_j \rangle = 1$ for each $\alpha_{i,}\, \alpha_j \in S$, i = j

(2) $\langle \alpha_{i,} \alpha_j \rangle = 0$ for each $\alpha_i, \alpha_j \in S$, i $\neq$ j

(OR)

The set S Subset of V(F) is an orthonormal set if $\langle \alpha_{i,} \alpha_j \rangle$ = 1for i = j and if $\langle \alpha_{i,} \alpha_j \rangle$= 0 for i$\neq$ j where $\alpha_i, \alpha_j \in S$.

That is $\langle \alpha_{i,} \alpha_j \rangle = \delta_{ij}$ (Kronecker delta)

NOTE 1. S $\subset V$ is an Orthonormal set $\Leftrightarrow$ S contains mutually orthogonal unit vectors.

2.An orthonormal set is an orthogonal set with the property that each vector is of length 1.

3.An orthonormal set does not contain zero vector.

Example 1. If $\alpha \epsilon V$ is a non -zero vector then $\frac{1}{||\alpha||}$ α is a unit vector. Therefore $\left\{\frac{1}{||\alpha||}\ \alpha\right\}$ is an orthonormal set. Thus ,every inner product space which is not a null space has an orthonormal subset.

Example 2.The standard basis of the inner product space R^3 is the set

$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$.

Then $\|e_1\| = 1, \|e_2\| = 1, \|e_3\| = 1$and $\langle e_{1,} e_2\rangle = 0, \langle e_2, e_3\rangle = 0, \langle e_3, e_1\rangle = 0$

Thus the standard basis forms an orthonormal set.

Corollary:

In an inner product space any ortho normal set of vectors is linearly independent.

Proof : let S be a finite or infinite orthonormal set of vectors in an inner product space.

Then $\alpha_i, \alpha_j \in S \Rightarrow \langle \alpha_{i,} \alpha_j\rangle = 1$ for i=j and

$\langle \alpha_{i,} \alpha_j\rangle = 0$ for $i \neq j$.

Let T= $\{\alpha_1, \alpha_2 \ldots\ldots\ldots\ldots \alpha_n\}$ be a finite subset of S having n distinct vectors.

Now we show that the set T is linearly independent.

Let $a_1, a_2, a_3, \ldots\ldots a_n \in$ F such that $a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n = 0$ ....(1)

Consider $\langle a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n, \alpha_i\rangle$

$= a_1\langle\alpha_1, \alpha_i\rangle + a_2\langle\alpha_2, \alpha_i\rangle + \ldots\ldots + a_i\langle\alpha_i, \alpha_i\rangle + \ldots\ldots + a_n\langle\alpha_n, \alpha_i\rangle$

$= 0 + 0 + \ldots\ldots\ldots + a_i\langle\alpha_i, \alpha_i\rangle + \ldots\ldots + 0$ (by the definition)

$= a_i$

$\therefore \langle a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots a_n\alpha_n, \alpha_i\rangle = a_i$

$\Rightarrow \langle 0, \alpha_i\rangle = a_i$, i=1,2,3,...... n

$\Rightarrow 0 = a_i$, i=1,2,3......n

$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0, \ldots\ldots a_n = 0$

$\therefore$ Eq (1) $\Rightarrow$ all scalars equal to zero.

$\therefore$ The set T is linearly independent.

Hence the super set S of T is linearly independent

$\therefore$ Any orthonormal set of vectors is linearly independent.

Corollary 2:

If S is an orthonormal set of vectors in an inner product space V(F),then

$\beta = a_1\alpha_1+a_2\alpha_2+......+a_n\alpha_n$ implies $a_i = \langle \beta,\alpha_i \rangle$ for i= 1,2,3,.........n.

Proof:

Let V(F) be an inner product space

Let S be an orthonormal set in V(F).

By the def we have for each $\alpha_i, \alpha_j \in S \Rightarrow \langle \alpha_i, \alpha_j \rangle = 1$ for i=j and $\langle \alpha_i, \alpha_j \rangle = 0$ for $i \neq j$.

Let $\beta \in V$ and $\beta = a_1\alpha_1+a_2\alpha_2+......+a_n\alpha_n$.

For each i=1,2,3,.......... n,

Consider $\langle \beta,\alpha_i \rangle = \langle a_1\alpha_1+a_2\alpha_2+......+a_n\alpha_n,\alpha_i \rangle$

$= a_1\langle \alpha_1,\alpha_i \rangle+a_2\langle \alpha_2,\alpha_i \rangle+......+a_i\langle \alpha_i,\alpha_i \rangle+.....+a_n\langle \alpha_n ,\alpha_i \rangle$

$= 0+0+.........+a_i\langle \alpha_i,\alpha_i \rangle+......+0$

$= a_i$

$\therefore a_i = \langle \beta,\alpha_i \rangle$ for i= 1,2,3,.........n.

Note : Let V(F) be an inner product space and $S = \{\alpha_1, \alpha_2 .... \alpha_n\}$ be an orthonormal subset of V. If $\beta \in L(S)$ then $\beta = \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle \alpha_i$

Theorem: If $\{\alpha_1, \alpha_2 .... \alpha_n\}$ is an orthonormal set in an inner product space V(F) and β∈V . Then $\gamma = \beta - \langle \beta,\alpha_1\rangle \alpha_1 - \langle \beta,\alpha_2 \rangle \alpha_2 - \ldots\ldots\ldots\ldots - \langle \beta,\alpha_n \rangle \alpha_n$

- is orthogonal to each of $\alpha_1, \alpha_2 .... \alpha_n$.

Proof:

Let V(F) be an inner product space.

Let S = $\{\alpha_1, \alpha_2 .... \alpha_n\}$ is an orthonormal set in V(F).

By def we have for each $\alpha_i, \alpha_j \in S \Rightarrow \langle \alpha_i, \alpha_j \rangle = 1$ for i=j and $\langle \alpha_i, \alpha_j \rangle = 0$ for $i \neq j$.

Let γ∈V such that $\gamma = \beta - \sum_{j=1}^{n} \langle \beta, \alpha_j \rangle \alpha_j$

For any i = 1,2,3,.........n.

Consider $\langle \gamma,\alpha_i \rangle) = \langle \beta - \sum_{j=1}^{n} \langle \beta, \alpha_j \rangle \alpha_j, \alpha_i \rangle$

$= \langle \beta, \alpha_i \rangle - \langle \sum_{j=1}^{n} \langle \beta, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$

$= \langle \beta, \alpha_i \rangle - \langle \beta, \alpha_i \rangle \langle \alpha_i, \alpha_i \rangle$ { since $\langle \alpha_i, \alpha_j \rangle = 0$ for i≠j }

$= \langle \beta, \alpha_i \rangle - \langle \beta, \alpha_i \rangle$ { since $\langle \alpha_i, \alpha_i \rangle = 1$ }

$= 0$

$\therefore \langle \gamma,\alpha_i \rangle = 0 \ \forall$ i= 1,2,3........... n.

$\therefore$ γ is orthogonal to each of $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots\ldots \alpha_n$

PROBLEMS.

1.Prove that the set S = $\left\{\left(\frac{1}{3},\frac{-2}{3},\frac{-2}{3}\right),\left(\frac{2}{3},\frac{-1}{3},\frac{2}{3}\right),\left(\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right)\right\}$ is an orthonormal set in R^3 with standard inner product.

Solution :

Given S = $\left\{\left(\frac{1}{3},\frac{-2}{3},\frac{-2}{3}\right),\left(\frac{2}{3},\frac{-1}{3},\frac{2}{3}\right),\left(\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right)\right\}$

Let α = $\left(\frac{1}{3},\frac{-2}{3},\frac{-2}{3}\right)$, β = $\left(\frac{2}{3},\frac{-1}{3},\frac{2}{3}\right)$ γ = $\left(\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right)$

Now $\langle \alpha,\alpha\rangle = \sqrt{\left(\frac{1}{9}\right)+\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)} = \sqrt{\frac{9}{9}} = 1.$

$\langle \beta,\beta\rangle = \sqrt{\left(\frac{4}{9}\right)+\left(\frac{1}{9}\right)+\left(\frac{4}{9}\right)} = \sqrt{\frac{9}{9}} = 1.$

$\langle \gamma,\gamma\rangle = \sqrt{\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)+\left(\frac{1}{9}\right)} = \sqrt{\frac{9}{9}} = 1.$

And $\langle \alpha,\beta\rangle = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right)+\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right)$

$= \frac{2}{9}+\frac{2}{9}+\frac{-4}{9}$

$\therefore \langle \alpha, \beta \rangle = 0$

$$\langle \beta, \gamma \rangle = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)$$

$$= \frac{4}{9} + \frac{-2}{9} + \frac{-2}{9}$$

$$= 0$$

$$\langle \gamma, \alpha \rangle = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{-1}{3}\right)\left(\frac{-2}{3}\right)$$

$$= \frac{2}{9} + \frac{-4}{9} + \frac{2}{9}$$

$$= 0.$$

$\therefore$ $S = \left\{ \left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right), \left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right) \right\}$ is an orthonormal set.

2. If S={(1,1,0), (1, −1,1), (−1,1,2)}is an orthogonal set in an inner product space R^3(R)then find the orthonormal set.

Solution:

Given S = $\{(1,1,0),(1,-1,1),(-1,1,2)\}$

Let $\alpha=(1,1,0)$, $\beta=(1,-1,1)$, $\gamma=(-1,1,2)$

$\|\alpha\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$\|\beta\| = \sqrt{1^2+-1^2+1^2} = \sqrt{3}$

$\|\gamma\| = \sqrt{-1^2+1^2+2^2} = \sqrt{6}$

$\therefore$ The orthonormal set of S = $\left\{\frac{\propto}{\|\alpha\|}, \frac{\beta}{\|\beta\|}, \frac{\gamma}{\|\gamma\|}\right\}$

$= \left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\right\}.$

3.If S = $\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)\right\}$ is an orthonormal sub set of an inner product space R^3(R), express the vector (2,1,3) as a linear combination of the basis vectors of S.

Solution:

Given S = $\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)\right\}$ is an orthonormal set in an inner product space R^3(R)

Let $\alpha_1 = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, $\alpha_2 = \left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\alpha_3 = \left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$

Given vector β = (2,1,3)

Let β = $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$ where $a_1, a_2, a_3 \in$ F .

Now $a_1 = \langle \beta, \alpha_1 \rangle = 2\left(\frac{1}{\sqrt{2}}\right) + 1\left(\frac{1}{\sqrt{2}}\right) + 3(0) = \frac{3}{\sqrt{2}}$

$a_2 = \langle \beta, \alpha_2 \rangle = 2\left(\frac{1}{\sqrt{3}}\right) + 1\left(\frac{-1}{\sqrt{3}}\right)) + 3\left(\frac{1}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$

$a_3 = \langle \beta, \alpha_3 \rangle = 2\left(\frac{-1}{\sqrt{6}}\right) + 1\left(\frac{1}{\sqrt{6}}\right) + 3\left(\frac{2}{\sqrt{6}}\right) = \frac{5}{\sqrt{6}}$

∴ β = $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

∴ β = $\frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) + \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + \frac{5}{\sqrt{6}}\left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$.

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